

KODAIRA DIMENSION OF IRREGULAR VARIETIES

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ABSTRACT. Let $f : X \rightarrow Y$ be an algebraic fiber space with general fiber F . If Y is of maximal Albanese dimension, we show that $\kappa(X) \geq \kappa(Y) + \kappa(F)$.

1. INTRODUCTION

Iitaka's $C_{n,m}$ conjecture states:

Conjecture 1.1. *Let $f : X \rightarrow Y$ be an algebraic fiber space with generic geometric fiber F , $\dim X = n$ and $\dim Y = m$. Then*

$$\kappa(X) \geq \kappa(Y) + \kappa(F).$$

This conjecture is of fundamental importance in the classification of higher dimensional complex projective varieties. It is known to hold in many important special cases; amongst these:

- (1) if $\kappa(Y) = \dim Y$, (cf. [Kawamata81] and [Viehweg82]),
- (2) if $\kappa(F) = \dim F$, (cf. [Kollár87]),
- (3) if F has a good minimal model (cf. [Kawamata85]), and
- (4) if $\dim Y = 1$, (cf. [Kawamata82]).

In this paper we will prove the following:

Theorem 1.2. *If Y is of maximal Albanese dimension then Conjecture 1.1 holds.*

Recall that by definition Y is of maximal Albanese dimension if and only if its Albanese morphism $Y \rightarrow \text{Alb}(Y)$ is generically finite. Note that if Y is a curve of non-negative Kodaira dimension (i.e. genus $g \geq 1$), then Y is of maximal Albanese dimension. Therefore, we obtain an independent proof of the result of [Kawamata82] mentioned above.

We also prove a conjecture of Ueno:

Theorem 1.3. *Let X be a smooth complex projective variety of Kodaira dimension 0. Then the Albanese morphism $a : X \rightarrow A$ is an algebraic fiber space (i.e. it is surjective with connected fibers) and if F is the general fiber of a , then $\kappa(F) = 0$.*

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Proof. The fact that $a : X \rightarrow A$ is an algebraic fiber space is shown in [Kawamata81]. By (1.2), we have $0 = \kappa(X) \geq \kappa(F) + \kappa(A) = \kappa(F)$. Since $\kappa(X) \geq 0$, we have $\kappa(F) \geq 0$ and so $\kappa(F) = 0$. \square

We remark that in [CH09] we showed a weaker version of (1.3), namely we showed that $\kappa(F) \leq \dim A$. In this paper we will make use of some of the techniques of [CH09] and in particular we will make use of multiplier ideal sheaf techniques, Kollár vanishing theorems and the Fourier-Mukai transform.

2. PRELIMINARIES

2.1. Notation. We work over the field of complex numbers \mathbb{C} .

Let X be a smooth complex projective variety, then the **Kodaira dimension** $\kappa(X) \in \{-\infty, 0, 1, \dots, \dim X\}$ is defined as follows: $\kappa(X) = -\infty$ if $|mK_X| = \emptyset$ for all $m > 0$ and otherwise

$$\kappa(X) = \max\{\dim \phi_{|mK_X|}(X) \mid m > 0\}.$$

Let $f : X \rightarrow Y$ be a projective morphism of normal varieties, then f is an **algebraic fiber space** if f is surjective with connected general geometric fiber $F = F_{X/Y}$.

If D is a divisor on a projective variety, then $\mathbf{B}(D) = \cap_{m>0} \text{Bs}(mD)$ denotes the **stable base locus** of D and if $V_i \subset |iD|$ is a sequence of linear series such that $V_i \cdot V_j \subset V_{i+j}$, then $\mathbf{B}(V_\bullet) = \cap_{m>0} \text{Bs}(V_m)$ is the **stable base locus** of V_\bullet .

2.2. Fourier Mukai transform. Let \mathcal{P} be the normalized Poincaré bundle on $A \times \hat{A}$ and $\hat{\mathcal{S}}$ be the functor from the category of \mathcal{O}_A -modules to that of $\mathcal{O}_{\hat{A}}$ -modules defined by

$$\hat{\mathcal{S}}(M) = \pi_{\hat{A},*}(\mathcal{P} \otimes \pi_A^* M).$$

By [Mukai81, 2.2] we have an isomorphisms of functors

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_A)^*[-g] \quad \text{and} \quad R\hat{\mathcal{S}} \circ R\mathcal{S} \cong (-1_{\hat{A}})^*[-g]$$

so that $R\mathcal{S}$ gives an equivalence of categories between $D(\hat{A})$ and $D(A)$ (and similarly for D^b , D_{qc}^- and D_c^-).

By [Mukai81, 3.1, 3.8] we have

$$\begin{aligned} \Delta_A \circ R\mathcal{S} &\cong ((-1_A)^* \circ R\mathcal{S} \circ \Delta_{\hat{A}})[g], & \text{and} \\ R\mathcal{S} \circ T_{\hat{x}}^* &\cong (\otimes P_{-\hat{x}}) \circ R\mathcal{S}, & R\mathcal{S} \circ (\otimes P_x) &\cong T_x^* \circ R\mathcal{S}. \end{aligned}$$

If L is an ample line bundle on \hat{A} , then there is an isogeny $\phi_L : \hat{A} \rightarrow A$ defined by $\phi_L(\hat{a}) = t_{\hat{a}}^* L^\vee \otimes L$. Then $\hat{L} = R^0 \mathcal{S}(L)$ is a vector bundle on A such that $\phi_L^*(\hat{L}^\vee) \cong L^{\oplus h}$ where $h = h^0(L)$. We will need the following:

Lemma 2.1. *Let \mathcal{F} be a quasi-coherent sheaf on A . Then*

$$D_k(R\Gamma(\mathcal{F} \otimes \hat{L}^\vee)) \cong R\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{F})) \otimes L).$$

Proof. This is an easy consequence of Groethendieck Duality and the projection formula (see the beginning of the proof of [Hacon04, 1.2]). \square

3. ADJOINT IDEALS

3.1. Adjoint ideals. The following definition is due to Takagi (cf. [Eisenstein10, 3.3]).

Definition 3.1. *Let $F \subset X$ be a smooth subvariety of codimension g in a smooth variety and D a \mathbb{Q} -divisor whose support does not contain F . Let $\mu : X' \rightarrow X$ be the blow up of X along F with reduced exceptional divisor E , $\nu : X'' \rightarrow X'$ a log resolution of $(X', \mu^{-1}(F) + \mu^{-1}(D))$ and let $\pi = \mu \circ \nu$. Then we define*

$$\mathrm{adj}_F(X, D) := \pi_* \mathcal{O}_{X''}([\mathcal{K}_{X''/X} - \pi^{-1}D - g \cdot \pi^{-1}F + \nu_*^{-1}E]).$$

Note that

$$\mathrm{adj}_F(X, D) \subset \mathcal{J}(X, D) := \pi_* \mathcal{O}_{X''}([\mathcal{K}_{X''/X} - \pi^{-1}D]).$$

By [Eisenstein10, 5.1] we have:

Theorem 3.2. *With the above notation, there is a short exact sequence*

$$0 \rightarrow \mathcal{J}(X, gF + D) \rightarrow \mathrm{adj}_F(X, D) \rightarrow \mathcal{J}(F, D|_F) \rightarrow 0.$$

Assume now that $a : X \rightarrow A$ is a morphism of smooth projective varieties $Z \subset A$ is a finite union of closed points with corresponding ideal $\mathfrak{m}_Z \subset \mathcal{O}_A$ and that $F = a^{-1}(Z) \subset X$ is a union of fibers.

Corollary 3.3. *Let H and L be Cartier divisors on A and D_0 be a Cartier divisor on X such that*

- (1) H is very ample and $\mathcal{O}_A(gH) \otimes \mathfrak{m}_Z$ is generated,
- (2) $D \sim_{\mathbb{Q}} D_0 + \epsilon a^*H$ for some $\epsilon \geq 0$, and
- (3) $L - (g + \epsilon)H$ is ample (resp. $L - (2g + \epsilon)H$ is ample),

then

$$H^i(A, a_*(\omega_X(D_0 + a^*L) \otimes \mathrm{adj}_F(X, D))) = 0 \quad \forall i > 0$$

(resp. $a_*(\omega_X(D_0 + a^*L) \otimes \mathrm{adj}_F(X, D))$ is generated).

Proof. Let $\mu : X' \rightarrow X$ be the blow up of F and let E be the exceptional divisor, then (cf. [Lazarsfeld04, 9.2.33])

$$\mathcal{J}(X, gF + D) = \mu_* (\mathcal{J}(X', gE + \mu^*D) \otimes \mathcal{O}_{X'}(\mathcal{K}_{X'/X})).$$

Let $G \sim_{\mathbb{Q}} gH$ be a general \mathbb{Q} -divisor vanishing along $Z \subset A$ to order g . Let $\nu : X'' \rightarrow X'$ be a log resolution of $(X', E + \mu^*D + \mu^*a^*G)$ and $\pi = \mu \circ \nu$, then $[\pi^*a^*G] = g\nu^*E$ and

$$\mathcal{J}(X, gF + D) = \pi_* \mathcal{O}_{X''}(\mathcal{K}_{X''/X} - [\pi^*(D + a^*G)]).$$

Since

$$\pi^*(D_0 + a^*L) - [\pi^*(D + a^*G)] \sim_{\mathbb{Q}} \{\pi^*(D + a^*G)\} + \pi^*a^*(L - (g + \epsilon)H),$$

it follows that by Kollár vanishing (cf. [Kollár95, 10.15]) that

$$R^i \pi_* \mathcal{O}_{X''}(K_{X''/X} + \pi^* D_0 - \lfloor \pi^*(D + a^* G) \rfloor) = 0 \quad \forall i > 0,$$

$R^i a_*(\omega_X(D_0) \otimes \mathcal{J}(X, gF + D)) = R^i(a \circ \pi)_* \mathcal{O}_{X''}(K_{X''} + \pi^* D_0 - \lfloor \pi^*(D + a^* G) \rfloor)$ is torsion free for all i and

$$H^i(A, a_*(\omega_X(D_0 + a^* L) \otimes \mathcal{J}(X, gF + D))) =$$

$$H^i(A, (a \circ \pi)_* \mathcal{O}_{X''}(K_{X''} + \pi^*(D_0 + a^* L) - \lfloor \pi^*(D + a^* G) \rfloor)) = 0$$

for all $i > 0$. Since $a_*(\omega_F(D_0|_F) \otimes \mathcal{J}(F, D|_F))$ is supported on the finite subset $Z \subset A$, it follows that we have a short exact sequence

$$\begin{aligned} 0 \rightarrow a_*(\omega_X(D_0 + a^* L) \otimes \mathcal{J}(X, gF + D)) \\ \rightarrow a_*(\omega_X(D_0 + a^* L) \otimes \text{adj}_F(X, D)) \\ \rightarrow a_*(\omega_F(D_0|_F) \otimes \mathcal{J}(F, D|_F)) \rightarrow 0. \end{aligned}$$

We also have

$$H^i(A, a_*(\omega_F(D_0|_F) \otimes \mathcal{J}(F, D|_F))) = 0 \quad \forall i > 0$$

and hence $H^i(A, a_*(\omega_X(D_0 + a^* L) \otimes \text{adj}_F(X, D))) = 0$ for $i > 0$ as required.

The remaining statement is a standard consequence of Castelnuovo-Mumford regularity (see eg. [Lazarsfeld04, 1.8.3]). \square

From now on we assume that $F \subset X$ is a general fiber of a morphism $a : X \rightarrow A$ from a smooth projective variety X to an abelian variety A . We may choose the origin $0 \in A$ so that $F = X_0$ is the fiber over 0.

If V is a linear series such that F is not contained in $\text{Bs}(V)$ and $D \in V$ is general, we define the ideal

$$\text{adj}_F(X, c \cdot V) = \text{adj}_F(X, c \cdot D).$$

Lemma 3.4. *With the above notation, we have*

$$\text{adj}_F(X, c \cdot V) \supset \text{adj}_F(X, c \cdot D'),$$

for any $D' \in V$ such that $F \not\subset \text{Supp}(D')$.

Proof. Standard. \square

If $V_i \subset |iL|$ is a sequence of linear series such that $V_i \cdot V_j \subset V_{i+j}$ and $F \not\subset \mathbf{B}(V_\bullet)$, then we let

$$\text{adj}_F(X, c \cdot \|V_\bullet\|) = \bigcup_{i>0} \text{adj}_F(X, \frac{c}{i} \cdot D_i)$$

where $D_i \in V_i$ is general. Recall that as X is Noetherian, we have $\text{adj}_F(X, c \cdot \|V_\bullet\|) = \text{adj}_F(X, \frac{c}{i} \cdot D_i)$ for all $i > 0$ sufficiently divisible.

Fix $m > 0$ and $D_F \in |mK_F|$. By a result of Viehweg (cf. [Viehweg83], [Viehweg95] and [Kollár95]), for any rational number $\epsilon > 0$, the map

$$|t(mK_X + \epsilon a^* H)| \rightarrow |tmK_F|$$

is surjective for all $t > 0$ sufficiently divisible. Let

$$V_{t,\epsilon,D_F} = \{G \in |t(mK_X + \epsilon a^*H)| \text{ s.t. } G|_F = tD_F\}.$$

Note that $V_{i,\epsilon,D_F} \cdot V_{j,\epsilon,D_F} \subset V_{i+j,\epsilon,D_F}$ and so we define the ideal

$$\text{adj}_F(X, ||V_{\bullet,\epsilon,D_F}||) := \bigcup_{t>0} \text{adj}_F(X, \frac{1}{t} \cdot V_{t,\epsilon,D_F}).$$

3.2. Descending chains of adjoint ideals.

Lemma 3.5. *For any $0 < \epsilon' < \epsilon$ we have*

$$\text{adj}_F(X, ||V_{\bullet,\epsilon',D_F}||) \subset \text{adj}_F(X, ||V_{\bullet,\epsilon,D_F}||).$$

Proof. We can pick $t > 0$ sufficiently divisible such that

$$\text{adj}_F(X, ||V_{\bullet,\epsilon,D_F}||) = \text{adj}_F(X, \frac{1}{t} \cdot V_{t,\epsilon,D_F}) \quad \text{and}$$

$$\text{adj}_F(X, ||V_{\bullet,\epsilon',D_F}||) = \text{adj}_F(X, \frac{1}{t} \cdot V_{t,\epsilon',D_F}).$$

Moreover, we have

$$V_{t,\epsilon',D_F} + |t(\epsilon - \epsilon')a^*H| \subset V_{t,\epsilon,D_F}.$$

Pick a general $G \in V_{t,\epsilon',D_F}$ so that $\text{adj}_F(X, \frac{1}{t} \cdot V_{t,\epsilon',D_F}) = \text{adj}_F(X, \frac{1}{t} \cdot G)$. If $G' \in |t(\epsilon - \epsilon')a^*H|$ is general, then $\text{adj}_F(X, \frac{1}{t} \cdot G) = \text{adj}_F(X, \frac{1}{t} \cdot (G + G'))$. By Lemma 3.4, we have

$$\text{adj}_F(X, \frac{1}{t} \cdot (G + G')) \subset \text{adj}_F(X, \frac{1}{t} \cdot V_{t,\epsilon,D_F}).$$

□

Lemma 3.6. *There exists a constant $0 < \epsilon_0 \ll 1$ such that*

$$a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet,\epsilon,D_F}||)) = a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet,\epsilon_0,D_F}||))$$

for all $\epsilon < \epsilon_0$. Moreover, if we denote this coherent sheaf by $\mathcal{F}_{||D_F||}$, then

- (1) $H^i(A, \mathcal{F}_{||D_F||} \otimes \mathcal{O}_A(L)) = 0$ if $i > 0$ and $L - gH$ is ample;
- (2) $\mathcal{F}_{||D_F||} \otimes \mathcal{O}_A(L)$ is generated if $L - 2gH$ is ample.

Proof. By (3.3), one sees that if $L - 2gH$ is ample, then there exists a rational number $0 < \epsilon_0 \ll 1$ such that $a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet,\epsilon,D_F}||)) \otimes \mathcal{O}_A(L)$ is generated for all $0 < \epsilon \leq \epsilon_0$. Therefore, if the inclusion

$$a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet,\epsilon',D_F}||)) \subset a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet,\epsilon,D_F}||))$$

is strict for some $0 < \epsilon' < \epsilon \leq \epsilon_0$, then

$$\begin{aligned} h^0(a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet,\epsilon',D_F}||)) \otimes \mathcal{O}_A(L)) \\ < h^0(a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet,\epsilon,D_F}||)) \otimes \mathcal{O}_A(L)). \end{aligned}$$

Since $H^0(A, a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet,\epsilon_0,D_F}||)) \otimes \mathcal{O}_A(L))$ is finite dimensional, we may assume that the above inclusions are equalities for any $0 < \epsilon' < \epsilon \leq \epsilon_0 \ll 1$.

The remaining statements then follow from (3.3). \square

Lemma 3.7. *The sheaf $\mathcal{F}_{||D_F||}$ is independent of H .*

Proof. Standard. \square

Convention. Fixed H and D_F as above, we may pick $0 < \epsilon \ll 1$, $t = t(\epsilon) \gg 0$ and $D \in |t(mK_X + a^*\epsilon H)|$ such that $\mathcal{F}_{||D_F||}$ is computed by (ϵ, t, D) , that is

$$\mathcal{F}_{||D_F||} = a_*(\omega_X^{\otimes m+1} \otimes \text{adj}_F(X, \frac{1}{t} \cdot D)).$$

3.3. Inductive limits. Let $\pi_k : A_k \rightarrow A$ be multiplication by 2^k so that $\pi_k = \pi_{k-1} \circ \pi_1$ and $A_k \cong A$. We let $X_k = X \times_A A_k$ and $\pi_{X,k} : X_k \rightarrow X$, $a_k : X_k \rightarrow A_k$. We have a sequence of inclusions of \mathcal{O}_A modules

$$\mathcal{O}_A \subset \pi_{1*}\mathcal{O}_{A_1} \subset \pi_{2*}\mathcal{O}_{A_2} \cdots \subset \mathcal{O}_\infty := \bigcup_{i \geq 0} \pi_{i*}\mathcal{O}_{A_i}.$$

Then \mathcal{O}_∞ is the direct sum of all topological trivial line bundles $P \in \text{Pic}^0(A)$ such that $\pi_k^*P = \mathcal{O}_{A_k}$ for some $k > 0$. We may think of A_∞ as the inverse limit of the system $\dots A_k \rightarrow A_{k-1} \rightarrow \dots$

We have the following diagram

$$\begin{array}{ccc} X_k & \xrightarrow{\pi_{X,k}} & X \\ \downarrow a_k & & \downarrow a \\ A_k & \xrightarrow{\pi_k} & A \end{array}$$

Fix once and for all a very ample divisor H on A , and let H_k be the corresponding divisor on A_k so that $\pi_k^*H \equiv 2^{2k}H_k$ (see for example [Mumford70, Section 6, Corollary 3]). Let $\mathcal{F}_{||D_F||}^k \in \text{Coh}(A_k)$ be the sheaves defined analogously to $\mathcal{F}_{||D_F||}$ (cf. (3.6)) so that

$$\mathcal{F}_{||D_F||}^k = a_{k*}(\omega_{X_k}^{\otimes m+1} \otimes \text{adj}_{F_k}(X_k, \frac{1}{t} \cdot D_k))$$

where $F_k = a_k^{-1}(0) \cong F$, and D_k is a general member of $V_{t,\epsilon,D_F}^k := |t(mK_{X_k} + a_k^*\epsilon H_k)|$ such that $D_k|_{F_k} = tD_F$, $0 < \epsilon \ll 1$ and $t \gg 0$.

Let $\mathcal{V}_{||D_F||}^k$ be the coherent sheaves on A_k defined using $\mathcal{J}(X_k, ||V_{\bullet,\epsilon,D_F}^k||)$ instead of $\text{adj}_{F_k}(X_k, ||V_{\bullet,\epsilon,D_F}^k||)$ so that

$$\mathcal{V}_{||D_F||}^k = a_{k*}(\omega_{X_k}^{\otimes m+1} \otimes \mathcal{J}(X_k, ||V_{\bullet,\epsilon,D_F}^k||)).$$

We now define coherent sheaves on A

$$\mathcal{G}_{||D_F||}^k := \pi_{k*}\mathcal{F}_{||D_F||}^k, \quad \mathcal{W}_{||D_F||}^k := \pi_{k*}\mathcal{V}_{||D_F||}^k.$$

Since $\text{adj}_{F_k}(X_k, ||V_{\bullet,\epsilon,D_F}^k||) \subset \mathcal{J}(X_k, ||V_{\bullet,\epsilon,D_F}^k||)$, we have $\mathcal{G}_{||D_F||}^k \subset \mathcal{W}_{||D_F||}^k$ for all $k > 0$.

Lemma 3.8. *For all $k > 0$ we have inclusions*

$$\mathcal{G}_{\|D_F\|}^k \subset \mathcal{G}_{\|D_F\|}^{k+1} \quad \text{and} \quad \mathcal{W}_{\|D_F\|}^k \subset \mathcal{W}_{\|D_F\|}^{k+1} \subset \mathcal{V}_{\|D_F\|} \otimes \pi_{k+1*} \mathcal{O}_{A_{k+1}}.$$

Proof. By induction on k , to prove the first inclusion, it suffices to prove that $\mathcal{F}_{\|D_F\|} \subset \pi_{1*} \mathcal{F}_{\|D_F\|}^1$.

We may assume that there exist ϵ_0, t_0 and $D \in |t_0(mK_X + a^* \epsilon_0 H)|$ such that $\mathcal{F}_{\|D_F\|}$ is computed by (ϵ_0, t_0, D) . Let $D_1 = \pi_{X,1}^* D$ on X_1 . It is clear that

$$D_1 \in \pi_{X,1}^* |t_0(mK_X + a^* \epsilon_0 H)| \subset |\pi_{X,1}^* (t_0(mK_X + a^* \epsilon_0 H))| = |t_0(mK_{X_1} + a_1^* \epsilon_0 \pi_1^* H)|.$$

We may furthermore assume that with this choice of ϵ_0 and t_0 , a general element $D'_1 \in |t_0(mK_{X_1} + a_1^* \epsilon_0 \pi_1^* H)|$ computes $\mathcal{F}_{\|D_F\|}^1$. We have

$$\begin{aligned} \text{adj}_{F_1}(X_1, \frac{1}{t} \cdot D'_1) &\supset \text{adj}_{F_1}(X_1, \frac{1}{t} \cdot D_1) \\ &\supset \text{adj}_{\pi_{X,1}^{-1}(F)}(X_1, \frac{1}{t} \cdot D_1) = \pi_{X,1}^* \text{adj}_F(X, \frac{1}{t} \cdot D). \end{aligned}$$

The desired inclusion now follows by pushing forward (via a) the inclusion

$$\begin{aligned} \omega_X^{\otimes m+1} \otimes \text{adj}_F(X, \frac{1}{t} \cdot D) &\subset \pi_{X,1*}(\omega_{X_1}^{\otimes m+1} \otimes \pi_{X,1}^* \text{adj}_F(X, \frac{1}{t} \cdot D)) \\ &\subset \pi_{X,1*}(\omega_{X_1}^{\otimes m+1} \otimes \text{adj}_{F_1}(X_1, \frac{1}{t} \cdot D'_1)). \end{aligned}$$

The proof of the inclusions $\mathcal{W}_{\|D_F\|}^k \subset \mathcal{W}_{\|D_F\|}^{k+1} \subset \mathcal{V}_{\|D_F\|} \otimes \pi_{k+1*} \mathcal{O}_{A_{k+1}}$ is similar. \square

4. PROOFS OF THEOREMS 1.2 AND 1.3

Lemma 4.1. *If $\kappa(X) = 0$ and $P_{m+1}(X) = 1$, then $\mathcal{V}_{\|D_F\|}^k$ is a unipotent vector bundle and for all $k > 0$ we have*

$$\mathcal{G}_{\|D_F\|}^k \subset \mathcal{W}_{\|D_F\|}^k = \mathcal{V}_{\|D_F\|} \otimes \pi_{k*} \mathcal{O}_{A_k}.$$

Proof. The inclusion $\mathcal{G}_{\|D_F\|}^k \subset \mathcal{W}_{\|D_F\|}^k$ has already been observed above. Since $X_k \rightarrow X$ is étale, $\kappa(X_k) = 0$ and hence $1 \geq P_{m+1}(X_k) \geq P_{m+1}(X) = 1$. By the proof of [Hacon04, 5.4], for all $k > 0$ we have that $\mathcal{V}_{\|D_F\|}^k$ is a unipotent vector bundle on A_k of rank r (where $0 < r \leq P_{m+1}(F)$). Hence $\mathcal{V}_{\|D_F\|}^k$ admits a filtration with successive quotients isomorphic to \mathcal{O}_{A_k} . Since $\pi_{k*} \mathcal{O}_{A_k} = \bigoplus_{P \in \text{Ker}(A_k \rightarrow \hat{A}_k)} P$ is a homogeneous vector bundle of rank 2^{2kg} , it follows that $\pi_{k*} \mathcal{V}_{\|D_F\|}^k$ is homogeneous of rank $2^{2kg}r$.

On the other hand, $\mathcal{V}_{\|D_F\|} \otimes \pi_{k*} \mathcal{O}_{A_k}$ has rank $2^{2kg}r$ and so the inclusion $\mathcal{W}_{\|D_F\|}^k \subset \mathcal{V}_{\|D_F\|} \otimes \pi_{k*} \mathcal{O}_{A_k}$ of homogeneous vector bundles of the same rank gives the required equality. \square

Corollary 4.2. *Let $p_k : A_k \rightarrow A_{k-1}$ be the induced morphism. Then the natural map $p_k^* \mathcal{V}_{||D_F||}^{k-1} \rightarrow \mathcal{V}_{||D_F||}^k$ is an isomorphism.*

Proof. Consider the isomorphism

$$p_{k*} p_k^* \mathcal{V}_{||D_F||}^{k-1} \cong \mathcal{V}_{||D_F||}^{k-1} \otimes p_{k*} \mathcal{O}_{A_k} \cong p_{k*} \mathcal{V}_{||D_F||}^k$$

where the first equality is given by the projection formula and last equality follows from (4.1). Since p_k is étale, we then have a homomorphism $p_k^* \mathcal{V}_{||D_F||}^{k-1} \rightarrow \mathcal{V}_{||D_F||}^k$ which is a isomomorphism of homogeneous vector bundles. \square

Remark 4.3. *In order to prove (1.3), we would like to argue as follows: Let*

$$\mathcal{G}_{||D_F||}^\infty = \bigcup_{k>0} \mathcal{G}_{||D_F||}^k \subset \mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty.$$

We will show (cf. (4.5)) that for any sufficiently ample line bundle L on \hat{A} , we have

$$H^i(A, \mathcal{G}_{||D_F||}^\infty \otimes \hat{L}^\vee) = 0 \quad \forall i > 0.$$

By (2.1), we have that

$$R\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L) \cong D_k(R\Gamma(\mathcal{G}_{||D_F||}^\infty) \otimes \hat{L}^\vee)$$

which (following the proof of [Hacon04, 1.2]) should imply that $R\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty))$ is a sheaf so that

$$V^0(\mathcal{G}_{||D_F||}^\infty) \supset V^1(\mathcal{G}_{||D_F||}^\infty) \supset \dots \supset V^g(\mathcal{G}_{||D_F||}^\infty).$$

But then as $\mathcal{G}_{||D_F||}^\infty \neq 0$, by section 2.2, we have $H^0(A, \mathcal{G}_{||D_F||}^\infty \otimes P) \neq 0$ for some $P \in \text{Pic}^0(A)$. In turn this implies that

$$H^0(X_k, \omega_{X_k}^{\otimes m+1} \otimes \text{adj}_{F_k}(X_k, \frac{1}{i} \cdot V_{i,\epsilon,D_F}^k) \otimes P) = H^0(A, \mathcal{G}_{||D_F||}^k \otimes P) \neq 0.$$

Let $0 \neq \sigma \in H^0(X_k, \omega_{X_k}^{\otimes m+1} \otimes \text{adj}_{F_k}(X_k, \frac{1}{i} \cdot V_{i,\epsilon,D_F}^k) \otimes P)$, then from the inclusion

$$\text{adj}_{F_k}(X_k, \frac{1}{i} \cdot V_{i,\epsilon,D_F}^k) \hookrightarrow \mathcal{J}(D_F) = \mathcal{O}_F(-D_F),$$

it follows that $\sigma|_F \in \omega_F^{\otimes m+1}(-D_F)$. If $\kappa(F) > 0$, then D_F varies in a positive dimensional family so that $\kappa(\omega_{X_k}^{\otimes m+1} \otimes P) > 0$. This in turn implies that $\kappa(X) > 0$. Therefore, if $\kappa(X) = 0$, we conclude that $\kappa(F) = 0$.

Lemma 4.4. *Let \mathcal{Q} be the cokernel of the inclusion $\mathcal{G}_{||D_F||}^\infty \hookrightarrow \mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty$. Then \mathcal{Q} is a coherent sheaf.*

Proof. Recall that there is an inclusion $\mathcal{F}_{||D_F||}^k \hookrightarrow \mathcal{V}_{||D_F||}^k$. Let \mathcal{R}_k be the cokernel of this inclusion. Since π_k is étale, $R^1 \pi_{k*} \mathcal{F}_{||D_F||}^k = 0$. Thus, pushing forward by π_k , we obtain short exact sequences

$$0 \rightarrow \mathcal{G}_{||D_F||}^k \rightarrow \mathcal{V}_{||D_F||} \otimes \mu_{k*} \mathcal{O}_{A_k} \rightarrow \mathcal{Q}_k \rightarrow 0.$$

Consider now the morphism $p_k : A_k \rightarrow A_{k-1}$. We have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & p_k^* \mathcal{F}_{||D_F||}^{k-1} & \longrightarrow & p_k^* \mathcal{V}_{||D_F||}^{k-1} & \longrightarrow & p_k^* \mathcal{R}_{k-1} \longrightarrow 0 \\ & & \alpha' \downarrow & & \beta' \downarrow & & \gamma' \downarrow \\ 0 & \longrightarrow & \mathcal{F}_{||D_F||}^k & \longrightarrow & \mathcal{V}_{||D_F||}^k & \longrightarrow & \mathcal{R}_k \longrightarrow 0. \end{array}$$

Notice that by the proof of (3.8) α' is inclusion and by (4.2) β' is an isomorphism. Therefore γ' is surjective. Pushing forward via p_k , one sees that the composition $\mathcal{R}_{k-1} \rightarrow p_k^* \mathcal{R}_{k-1} \rightarrow \mathcal{R}_k$ is surjective. Therefore, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_{||D_F||}^{k-1} & \longrightarrow & \mathcal{V}_{||D_F||}^{k-1} & \longrightarrow & \mathcal{R}_{k-1} \longrightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \longrightarrow & p_{k*} \mathcal{F}_{||D_F||}^k & \longrightarrow & p_{k*} \mathcal{V}_{||D_F||}^k & \longrightarrow & p_{k*} \mathcal{R}_k \longrightarrow 0 \end{array}$$

where γ is surjective. Pushing forward via π_{k-1} , we then have the surjection $\phi_{k-1} : \mathcal{Q}_{k-1} \rightarrow \mathcal{Q}_k$ for all $k > 0$. Since \mathcal{Q}_k has finite length (for all k), we have that ϕ_k is an isomorphism for $k \gg 0$. Let $\mathcal{Q} = \mathcal{Q}_k$ for $k \gg 0$ be the resulting coherent sheaf. Taking the direct limit of the push-forward of the above diagram, we thus complete the proof. \square

Lemma 4.5. *If L is any sufficiently ample line bundle on \hat{A} , then $H^i(A, \mathcal{G}_{||D_F||}^\infty \otimes \hat{L}^\vee) = 0$ for $i > 0$.*

Proof. By [Hartshorne77, III.2.9], since $\mathcal{G}_{||D_F||}^\infty = \varinjlim \mathcal{G}_{||D_F||}^k$, then

$$H^i(A, \mathcal{G}_{||D_F||}^\infty \otimes \hat{L}^\vee) = \varinjlim H^i(A, \mathcal{G}_{||D_F||}^k \otimes \hat{L}^\vee).$$

It suffices therefore to check that for any L , we have $H^i(A, \mathcal{G}_{||D_F||}^k \otimes \hat{L}^\vee) = 0$ for $i > 0$ and $k \gg 0$. Let $\phi_L : \hat{A} \rightarrow A$, then $\phi_L^* \hat{L}^\vee \cong L^{\oplus h}$ where $h = h^0(L)$, and so \hat{L}^\vee is an ample vector bundle. If $A'_k = A_k \times_A \hat{A}$, then we have the following diagram:

$$\begin{array}{ccc} A'_k & \xrightarrow{\psi} & \hat{A} \\ \rho \downarrow & & \downarrow \phi_L \\ A_k & \xrightarrow{\pi_k} & A \end{array}$$

Notice that:

- (1) $\deg(\rho) = h$.
- (2) $\rho^* \pi_k^* \hat{L}^\vee = \psi^* \phi_L^* \hat{L}^\vee = \psi^* L^{\oplus h}$ is a direct sum of line bundles.
- (3) Since $\rho^* H_k$ is an ample line bundle, $3\rho^* H_k$ is very ample.
- (4) Since $3H_k \otimes \mathbf{m}_0$ is generated, so is $3g\rho^* H_k \otimes \mathbf{m}_{\rho^{-1}(0)}$.
- (5) Since $\hat{L}^\vee - \delta H$ is ample for some $\delta > 0$ and $\pi_k^* H \equiv 2^{2k} H_k$, it follows that $\pi_k^* \hat{L}^\vee \otimes \mathcal{O}_{A_k}(-3gH_k)$ is ample for $k \gg 0$. Thus $\rho^*(\pi_k^* \hat{L}^\vee \otimes \mathcal{O}_{A_k}(-3gH_k))$ is a direct sum of ample line bundles.

Let $X'_k = X_k \times_{A_k} A'_k$, $a'_k : X'_k \rightarrow A'_k$ and $F'_k \subset X'_k$ be the inverse image of F_k (so that $F'_k = a'^{-1}_k(\rho^{-1}(0))$ has h distinct irreducible components isomorphic to F_k). It is easy to see that

$$\rho^*(\mathcal{F}^k_{||D_F||}) = a'_{k*}(\omega^{\otimes m+1}_{X'_k} \otimes \text{adj}_{F'_k}(X'_k, \frac{1}{t} \cdot D'_k)),$$

where $D'_k \in |tm(K_{X'_k} + a'^*_k \rho^* \epsilon H_k)|$ is the pull-back of a general element $D_k \in |tm(K_{X_k} + a^*_k \epsilon H_k)|$ such that $D_k|_{F_k} = tD_F$, $0 < \epsilon \ll 1$ and $t \gg 0$. Since $\frac{1}{t} \cdot D'_k \sim_{\mathbb{Q}} m(K_{X'_k} + (\rho \circ a'_k)^* \epsilon H_k)$ and $\pi_k^* \hat{L}^\vee \otimes \mathcal{O}_{A_k}(-(3g + m\epsilon)H_k)$ is ample for $\epsilon \ll 0$, by (3.3), one sees that

$$H^i(A'_k, \rho^*(\mathcal{F}^k_{||D_F||} \otimes \pi_k^* \hat{L}^\vee)) = H^i(A'_k, a'_{k*}(\omega^{\otimes m+1}_{X'_k} \otimes \text{adj}_{F'_k}(X'_k, \frac{1}{t} \cdot D'_k)) \otimes \rho^* \pi_k^* \hat{L}^\vee) = 0$$

for all $i > 0$. Since ρ is étale, by the projection formula we have that

$$H^i(A_k, \mathcal{F}^k_{||D_F||} \otimes \pi_k^* \hat{L}^\vee) = 0 \quad \forall i > 0.$$

Since π_k is étale, and $\pi_{k*} \mathcal{F}_{||D_F||} = \mathcal{G}^k_{||D_F||}$, it now follows easily that $H^i(A, \mathcal{G}^k_{||D_F||} \otimes \hat{L}^\vee) = 0$ for $i > 0$ and $k \gg 0$. \square

Lemma 4.6. *We have $R^i \hat{\mathcal{S}}(D_A(\mathcal{G}^\infty_{||D_F||})) = 0$ for $i \neq g$.*

Proof. Since $\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty = \bigoplus_{P \in \text{Ker}(\hat{\pi}_k)} \mathcal{V}_{||D_F||} \otimes P$, then (cf. Subsection 2.2)

$$\begin{aligned} R\hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty)) &= R^g \hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty)) = \\ \bigoplus_{P \in \text{Ker}(\hat{\pi}_k)} R^g \hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes P)) &= \bigoplus_{P \in \text{Ker}(\hat{\pi}_k)} \widehat{\mathcal{V}_{||D_F||}^\vee} \otimes P^\vee = \bigoplus_{x \in \text{Ker}(\pi_k)} T_x^*(\widehat{\mathcal{V}_{||D_F||}^\vee}). \end{aligned}$$

Therefore, by (4.4), we get an exact sequence

$$\begin{aligned} 0 \rightarrow R^{g-1} \hat{\mathcal{S}}(D_A(\mathcal{G}^\infty_{||D_F||})) \rightarrow \widehat{D_A(\mathcal{Q})} \rightarrow R^g \hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty)) \\ \rightarrow R^g \hat{\mathcal{S}}(D_A(\mathcal{G}^\infty_{||D_F||})) \rightarrow 0. \end{aligned}$$

In particular, $R^i \hat{\mathcal{S}}(D_A(\mathcal{G}^\infty_{||D_F||})) = 0$ for $i \notin \{g-1, g\}$ and $\widehat{D_A(\mathcal{Q})} \rightarrow R^g \hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty))$ is a homomorphism from a unipotent vector bundle to a direct sum of Artinian modules. Let

$$\tau = \text{Im} \left(\widehat{D_A(\mathcal{Q})} \rightarrow R^g \hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty)) \right)$$

and consider the induced short exact sequence

$$0 \rightarrow \tau \rightarrow R^g \hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty)) \rightarrow R^g \hat{\mathcal{S}}(D_A(\mathcal{G}^\infty_{||D_F||})) \rightarrow 0.$$

Since $\tau \subset R^g \hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty))$ are direct sums of Artinian modules, so is $R^g \hat{\mathcal{S}}(D_A(\mathcal{G}^\infty_{||D_F||}))$. This implies in particular $H^i(\hat{A}, R^g \hat{\mathcal{S}}(D_A(\mathcal{G}^\infty_{||D_F||})) \otimes L) = 0$ for $i > 0$ and any line bundle L .

Since $R^{g-1}\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \subset \widehat{D_A(\mathcal{Q})}$ is a coherent sheaf, we then have that $R^{g-1}\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L$ is generated and $H^i(\hat{A}, R^{g-1}\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L) = 0$ for $i > 0$, and L sufficiently ample.

Following the proof of [Hacon04, 1.2], we consider the spectral sequence

$$E_2^{i,j} = R^i\Gamma(R^j\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L)$$

abutting to $R^{i+j}\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L)$. Since $E_2^{i,j} = 0$ if $(i,j) \notin \{(0, g-1), (0, g)\}$, it follows that the above spectral sequence degenerates at the E_2 term.

By (2.1), we have that

$$D_k(R\Gamma(\mathcal{G}_{||D_F||}^\infty \otimes \hat{L}^\vee)) \cong R\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L).$$

By (4.5), we have that $R\Gamma(\mathcal{G}_{||D_F||}^\infty \otimes \hat{L}^\vee) = R^0\Gamma(\mathcal{G}_{||D_F||}^\infty \otimes \hat{L}^\vee)$. Therefore, $R^{g-1}\Gamma(R\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L) = 0$ and hence $E_2^{0,g-1} = 0$.

But then $R^0\Gamma(R^{g-1}\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L) = 0$ so that $R^{g-1}\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \otimes L = 0$. \square

Corollary 4.7. $\mathcal{G}_{||D_F||}^\infty \cong \mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty$.

Proof. Since $\widehat{D_A(\mathcal{Q})} \rightarrow R^g\hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty))$ is an injection from a vector bundle to a direct sum of torsion sheaves, $\widehat{D_A(\mathcal{Q})} = 0$. Thus $R^g\hat{\mathcal{S}}(D_A(\mathcal{G}_{||D_F||}^\infty)) \cong R^g\hat{\mathcal{S}}(D_A(\mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty))$ and the claim easily follows cf. Section (2.2). \square

We will now prove Theorem 1.3:

Theorem 4.8. *Let $a : X \rightarrow A$ be an algebraic fiber space, F the general fiber and A an abelian variety. If $\kappa(X) = 0$, then $\kappa(F) = 0$.*

Proof. Since $\kappa(X) = 0$, $S_t := \{P \in \text{Pic}^0(X) | h^0(\omega_X^{\otimes t} \otimes P) \neq 0\} = \{\mathcal{O}_X\}$ for all $t > 0$ sufficiently divisible cf. [CH02, 3.2]. Since $\mathcal{V}_{||D_F||} \subset a_*(\omega_X^{\otimes m+1})$ is a unipotent vector bundle, then for any $m+1$ sufficiently divisible, we have that $H^0(A, \mathcal{G}_{||D_F||}^\infty) \cong H^0(A, \mathcal{V}_{||D_F||} \otimes \mathcal{O}_\infty) \neq 0$. Since

$$H^0(A, \mathcal{G}_{||D_F||}^\infty) = \varinjlim H^0(A, \mathcal{G}_{||D_F||}^k),$$

we have that

$$H^0(A, \mathcal{G}_{||D_F||}^k) \neq 0 \quad \forall k \gg 0.$$

Pick any k such that

$$H^0(X_k, \omega_{X_k}^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet, \epsilon, D_F}||)) = H^0(A_k, \mathcal{F}_{||D_F||}^k) = H^0(A, \mathcal{G}_{||D_F||}^k) \neq 0.$$

Since $\kappa(X) = 0$, we have $\kappa(X_k) = 0$ and hence $H^0(X_k, \omega_{X_k}^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet, \epsilon, D_F}||)) \cong \mathbb{C}$. Let $0 \neq \sigma \in H^0(X_k, \omega_{X_k}^{\otimes m+1} \otimes \text{adj}_F(X, ||V_{\bullet, \epsilon, D_F}||))$, then as the homomorphism $\text{adj}_F(X, ||V_{\bullet, \epsilon, D_F}||) \rightarrow \mathcal{J}(F, D_F)$ is surjective, $\sigma|_F$ vanishes along D_F . If $\kappa(F) > 0$, then we may assume

that D_F varies in a positive dimensional family $|D_F|$ and hence that $h^0(X_k, \omega_{X_k}^{\otimes m+1}) > 0$. This is a contradiction and hence $\kappa(F) = 0$. \square

We will now prove Theorem 1.2:

Theorem 4.9. *Let $f : X \rightarrow Y$ be an algebraic fiber space with Y of maximal Albanese dimension and general fiber F . Then $\kappa(X) \geq \kappa(Y) + \kappa(F)$.*

Proof. Assume that $\kappa(X) < 0$, and let $a_X : X \rightarrow A_X$ (resp. $a_Y : Y \rightarrow A_Y$) be the Albanese morphism of X (resp. of Y). Let $\alpha_X : X \rightarrow X'$ (resp. $\alpha_Y : Y \rightarrow Y'$) be the Stein factorization of a_X (resp. a_Y). Since $f_*\mathcal{O}_X = \mathcal{O}_Y$, it follows that $X \rightarrow Y'$ is the Stein factorization of $X \rightarrow A_Y$. But then $f' : X' \rightarrow Y'$ is an algebraic fiber space and the following diagram is commutative

$$\begin{array}{ccccc} X & \xrightarrow{\alpha_X} & X' & \xrightarrow{\beta_X} & A_X \\ f \downarrow & & f' \downarrow & & \downarrow \\ Y & \xrightarrow{\alpha_Y} & Y' & \xrightarrow{\beta_Y} & A_Y. \end{array}$$

Since Y is of maximal Albanese dimension, α_Y is birational. Let G be the general fiber of $\alpha_X : X \rightarrow X'$ and E be the general fiber of $X' \rightarrow Y'$. Then F maps surjectively on to E with general fiber G . By [Lai09, 3.1] (applied to $X \rightarrow X'$), one sees that $\kappa(G) < 0$. By the easy addition formula (applied to $F \rightarrow E$), one sees that $\kappa(F) < 0$. Thus the Theorem is true if $\kappa(X) < 0$.

We may therefore assume that $\kappa(X) \geq 0$. We will follow the arguments of [Mori85, 6.4].

We will first show that the statement holds when $\kappa(Y) = 0$. In this case, by Kawamata's Theorem (cf. [Kawamata81]), we may assume that Y is an abelian variety. Let G be the general fiber of $\phi : X \rightarrow Z$, the Iitaka fibration of X , then $\kappa(G) = 0$. Let $K = f(G)$, then K is an abelian subvariety of Y and $f|_G : G \rightarrow K$ is the Albanese map of G . The general fiber of $f|_G$ is $F' := F \cap G$. By (4.8), we have $\kappa(F') = 0$. On the other hand, F' can be the general fiber of $\phi_F : F \rightarrow Z$. By easy addition, we have

$$\kappa(F) \leq \kappa(F') + \dim(\text{im}(\phi_F)) \leq \dim Z = \kappa(X).$$

We now consider the general case. Let $Y \rightarrow A_Y$ be the Albanese map of Y and $Y' \rightarrow A_Y$ be the Stein factorization of $Y \rightarrow A_Y$. Then $Y \rightarrow Y'$ is birational. We may thus replace Y by Y' and assume that $Y \rightarrow A_Y$ is finite. By [Kawamata81, Theorem 13], after replacing X and Y by étale covers, we may assume that there is an abelian variety K and a variety of general type W such that Y is isomorphic to $W \times K$. Note that $\kappa(W) = \dim W = \kappa(Y)$. Let E be the general fiber of the

algebraic fiber space $X \rightarrow W$. By [Kawamata81, Theorem 3], one has

$$\kappa(X) \geq \kappa(E) + \kappa(W) = \kappa(E) + \kappa(Y).$$

Notice also that $f|_E : E \rightarrow K$ has general fiber F . Since $\kappa(K) = 0$, by what we have shown above, we have

$$\kappa(E) \geq \kappa(F) + \kappa(K) = \kappa(F).$$

Hence we have

$$\kappa(X) \geq \kappa(Y) + \kappa(F),$$

as required. □

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